Journal of Advances in Mathematics and Computer Science

23(6): 1-11, 2017; Article no.JAMCS.35982

Previously known as British Journal of Mathematics & Computer Science
ISSN: 2231-0851

A Note on the Fuglede and Fuglede-Putnam's Theorems

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/JAMCS/2017/35982
Editor(s):
(1) Morteza Seddighin, Indiana University East Richmond, USA.
Reviewers:
(1) Xiaolan Liu, Sichuan University of Science and Engineering, China.
(2) Francisco Bulnes, Tecnológico de Estudios Superiores de Chalco, Mexico.
Complete Peer review History: http://www.sciencedomain.org/review-history/20570

Received: 6th August 2017
Accepted: 14th August 2017
Published: 19th August 2017

Original Research Article

Abstract

In this paper, we investigate the extension of Fuglede and Fuglede-Putnam's Theorems to two bounded linear operators.

Keywords: Fuglede's theorem; Fuglede-Putnam's theorem; Embry's theorem and normal operators.

AMS (2010): 47Axx.

1 Introduction

The Fuglede–Putnam theorem plays a major role in the theory of bounded operators. Many authors have worked on it since the papers Ahmed Bachir and M. Wahideddin Altanji [1] generalized that theorem for (p,q) quasiposinormal operators, Mohammed Hichem Mortad [2] generalized this theorem to isometry and co-isometry operators, M. H. M. Rashid [3] generalized this theorem by using Aluthge transform, Vasile Lauric [4] generalized this theorem for almost normal operators with finite modulus of Hilbert-Schmidt quasi-triangularity, Mahmood Kamil Shihab [5] proved some properties of square-normal operators by using this theorem and A. Ber, V. Chiln, F. Sukochev and D. Zanin [6] extended that theorem from the algebra B(H) of all bounded operators on the Hilbert space H to the algebra of all locally measurable operators affiliated with a von Neumann algebra.

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First, we notice that while we will be recalling most of the essential background we will assume the reader is familiar with any other result or notion which will appear in the present paper. Some of the standard textbooks on bounded operator theory are [7,8].

Second, in this section we present some of the preliminary results of Fuglede and Fuglede-Putnam theorems. Then in Section 3 and Section 4 we extend their results to two bounded linear operators. Then in Section 5 we extend Embry's Theorem and some corollaries. Fuglede [9] proved his theorem for one bounded linear operators as follows:

Theorem 1: *On a Hilbert space* H *let N be a normal operator, for any bounded linear operator A if $AN = NA$ then $AN^* = N^*A$.*

Putnam [10] generalized Theorem 1 to two normal operators as follows:

Theorem 2: *On a Hilbert space* H*, if N and M are normal operators on* H *and if A is a bounded operator on* H *such that $AN = MA$ then $AN^* = M^*A$.*

In the entire paper A,B,N and M represent continuous linear operators on a Hilbert space H. A^* is the adjoint of A, we said that A is normal if $A^*A = AA^*$, self-adjoint if $A = A^*$, unitary if $AA^* = A^*A = I$ where I is an identity operator and positive if $\langle Ax, x\rangle > 0$, σ (A) denotes the spectrum of A and the numerical range W(A) is the image of the unit sphere of H under the quadratic form $x \to \langle A_x, x\rangle$ associated with the operator. More precisely, $W(A) = \{\langle A_x, x\rangle : x \in H , \| x \| = 1\}$.

2 Extension of Fuglede's Theorem

In this section we present our results by extending the previous theorem of Fuglede to two bounded operators.

Theorem 3: *Let* H *be a Hilbert space, let N be a normal operator on* H *for any two bounded operators A and B on* H *if*

$$AN = NB \tag{1}$$

and

$$BN = NA \tag{2}$$

then

$$AN^* = N^*B \;\; \& \;\; BN^* = N^*A \tag{3}$$

Proof: By adding (1) and (2) we get

$$AN + BN = NB + NA$$

$$(A + B)N = N(A + B).$$

So by Fuglede's Theorem we have

$$(A + B)N^* = N^*(A + B)$$

$$AN^* + BN^* = N^*A + N^*B \tag{4}$$

$$AN^* - N^*B = N^*A - BN^*. \tag{5}$$

By subtracting (2) from (1) we get

$$AN - BN = NB - NA$$

$$(A - B)N = N(B - A)$$

$$(A - B)N = -N(A - B)$$

by Fuglede-Putnam's Theorem we have

$$(A - B)N^* = -N^*(A - B)$$

$$AN^* - BN^* = -N^*A + N^*B$$

$$AN^* - N^*B = BN^* - N^*A. \qquad (6)$$

By adding (5) with (6) we get

$$2AN^* - 2N^*B = 0$$

so

$$AN^* = N^*B.$$

By subtracting (6) from (5) we get

$$0 = -2BN^* + 2N^*A$$

so

$$BN^* = N^*A.$$

So (3) is proved.

We can prove (3) (another proof) by using trick matrix as follows:

Let $X = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ Y= $\begin{bmatrix} 0 & N \\ N & 0 \end{bmatrix}$ so Y* = $\begin{bmatrix} 0 & N^* \\ N^* & 0 \end{bmatrix}$

since Y is normal.

Also we have

$$XY = \begin{bmatrix} 0 & AN \\ BN & 0 \end{bmatrix}, \quad YX = \begin{bmatrix} 0 & NB \\ NA & 0 \end{bmatrix}.$$

So

$$XY = Y X.$$

So by Fuglede's Theorem we have

$$XY^* = Y^* X.$$

$$XY^* = \begin{bmatrix} 0 & AN^* \\ BN^* & 0 \end{bmatrix} \quad , \quad Y^*X = \begin{bmatrix} 0 & N^*B \\ N^*B & 0 \end{bmatrix}.$$

So we have

$$AN^* = N^*B \quad \& \quad BN^* = N^*A.$$

Now we give the following negative result:

Conjecture 1: *Let H be a Hilbert space and N be a normal operator, for any two bounded linear operators A and B, if*

$$AN = NB$$

then

$$BN^* = N^*A$$

This conjecture is not true, we give the following counter example:

Example 1: *Let*

$$A=\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad N = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2-i & 2+2i \\ 1+i & -1+2i \end{bmatrix} \quad N^* = \begin{bmatrix} 1 & i \\ -i & 2 \end{bmatrix}$$

Since N is not only normal but even self-adjoint.

$$AN = \begin{bmatrix} 1 & i \\ 1 & 2i \end{bmatrix} \quad , \quad NB = \begin{bmatrix} 1 & i \\ 1 & 2i \end{bmatrix}.$$

So

$$AN = NB.$$

But $BN^* = \begin{bmatrix} 4-3i & 5+6i \\ 3+2i & -3+5i \end{bmatrix}$, $N^*A = \begin{bmatrix} 1 & -1 \\ -i & 2i \end{bmatrix}$ so

$$BN^* \neq N^*A .$$

3 Extension of Fuglede-Putnam's Theorem

In this section we present our results by extending the previous theorem of Fuglede-Putnam to two bounded operators.

Theorem 4: *Let* H *be a Hilbert space, let N and M be normal operators on* H *for any two bounded operators A and B on* H *if*

$$AN = MB \tag{7}$$

$$BN = MA \tag{8}$$

then

$$AN^* = M^*B \, , \; BN^* = M^*A \qquad (9)$$

Proof. The proof is similar to the proof of Theorem 3.

We give the negative result to the conjecture that appeared in [2]

Conjecture 2: *Let H be a Hilbert space, let N and M be normal operators on H for any two bounded linear operators A and B on H if*

$$AN = MB$$

then

$$AN^* = M^*B.$$

This conjecture is not true, we give the following counter example:

Example 2: *Let*

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \; N = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}, \; M = \begin{bmatrix} -i & i \\ -i & -i \end{bmatrix}, \quad B = \begin{bmatrix} \frac{-1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{-1}{2} \end{bmatrix}$$

We have

$$N^* = \begin{bmatrix} -i & i \\ i & -i \end{bmatrix}, \quad M^* = \begin{bmatrix} i & i \\ -i & i \end{bmatrix}$$

Note that N and M are normal operators on H, we have

$$AN^* = \begin{bmatrix} i & -i \\ 0 & 0 \end{bmatrix}, \quad M^*B = \begin{bmatrix} i & -i \\ 0 & 0 \end{bmatrix}$$

so

$$AN = MB.$$

But

$$AN^* = \begin{bmatrix} -i & i \\ 0 & 0 \end{bmatrix}, \quad M^*A = \begin{bmatrix} 0 & 0 \\ i & -i \end{bmatrix}$$

so

$$AN^* \neq M^*B.$$

We give the negative result to our conjecture that is:

Conjecture 3: *Let H be a Hilbert space, let N and M be normal operators on H for any two bounded linear operators A and B on H if*

$$AN = MB$$

then

$$BN^* = M^*A.$$

This conjecture is not true, we give the following counter example:

Example 3: *From above Example 2 we have*

$$BN^* = \begin{bmatrix} i & -i \\ -i & i \end{bmatrix}, \quad M^*A = \begin{bmatrix} 2i & i \\ 0 & i \end{bmatrix}$$

So

$$BN^* \neq M^*A$$

4 Extension of Embry's Theorem

In this section we present Embry's Theorem and some of it's corollaries appeared in [11] that will be used in our proof:

Theorem 5: *If N and M are commuting normal operators and AN = MA, where 0 is not in the numerical range of A, then N = M.*

Corollary 1: *If A is an operator for which either* 0 doesn't belong W(A) *or* $\sigma(A) \cap \sigma(-A) = \emptyset$ *and* $AE = -EA$, *where either A or E is normal, then* $E = 0$.

Corollary 2: *If* $AE = E^*A$ *and* $AE^* = EA$, *where either* 0 doesn't belong to W(A) or $\sigma(A) \cap \sigma(-A) = \emptyset$, *then E is self-adjoint.*

Corollary 3: *If* $AE = E^*A$, *where either* 0 doesn't belong to W(A) *or* $\sigma(A) \cap \sigma(-A) = \emptyset$ *and either A is unitary or E is normal, then E is self-adjoint.*

Now we extend the previous theorem of Embry to two bounded operators.

Theorem 6: *If N and M are commuting normal operators and*

$$AN = MB \;\&\; BN = MA, \tag{10}$$

where 0 doesn't belong to W($A + B$), *then* $N = M$.

Proof. By adding two equations in (10) we have:

$$AN + BN = MB + MA$$

$$(A + B)N = M(A + B)$$

N and M commuting normal operators and 0 doesn't belong to W(A + B), so Theorem 5 is applicable, resulting in $N = M$.

Both conditions on above theorem (N, M are commutative and 0 doesn't belong to W(A + B)) are necessary, if one of them is failed then the theorem is not satisfied as in the following two examples.

Example 4: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad N = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}, \quad M = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 0 \\ 0 & i \end{bmatrix}$$

Since

$$AN = MB \;\&\; BN = MA,$$

also $NM = MN$ *but* 0 doesn't belong to W(A + B)*, so* $N \neq M$.

Example5: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad N = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \quad M = \begin{bmatrix} 1 & -2i \\ i & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$$

since

$$AN = MB \;\&\; BN = MA,$$

also 0 doesn't belong to W(A + B)*, but* $NM \neq MN$*, so* $N \neq M$.

Theorem 7: *For any operator A for which* $\sigma(A) \cap \sigma(-A) = \emptyset$ *if*

$$A\,N = MA \;\&\; AM = NA \tag{11}$$

then N = M, where N and M are normal operators.

Proof: s By subtracting two equations in (11) we get

$$AN - AM = MA - NA$$

$$A(N - M) = (M - N)A$$

$$A(N - M) = -(N - M)A$$

and also we have $\sigma(A) \cap \sigma(-A) = \emptyset$, so Corollary 1 is applicable, resulting in $N - M = 0$, so $N = M$.

Theorem 8: *Let A be a bounded linear operator such that* 0 doesn't belong to W(A)*, let B be a normal operator. If* $AB = B^*A$ *then B is self-adjoint.*

Proof. Since B is normal and so is B^* , also $BB^* = B^*B$, and $AB = B^*A$, where 0 doesn't belong to W(A), so Embry Theorem is applicable resulting in $B = B^*$ hence B is self-adjoint.

Corollary 4: *Let* 0 doesn't belong to W(A + B) *and either A or B is commutative with E. If*

$$AE = -EB \tag{12}$$

and

$$BE = -EA, \tag{13}$$

where either E or A&B are normal, then E = 0.

Proof. If E is normal then Theorem 6 is applicable, and we have $E = -E$ i.e

$$E = 0.$$

If A and B are normal then Theorem 4 is applicable, and we have

$$A^*E = -EB^* \ \& \ B^*E = -EA^*.$$

Take conjugate for the last two equations

$$E^*A = -BE^* \tag{14}$$

$$E^*B = -AE^*. \tag{15}$$

Now from two equations (12) and (13) we have if E is commutative with A then E is also commutative with B, and if E is commutative with B then E is also commutative with A.

Now by subtracting (13) from (15) we get

$$E^*B - BE = EA - AE^*$$

$$E^*B - EB = AE - AE^*$$

$$(E^* - E)B = A(E - E^*)$$

$$A(E - E^*) = -(E - E^*)B \tag{16}$$

Now by subtracting (12) from (14) we get

$$E^*A - AE = EB - BE^*$$

$$E^*A - EA = BE - BE^*$$

$$(E^* - E)A = B(E - E^*)$$

$$B(E - E^*) = -(E - E^*)A. \tag{17}$$

Satisfy Theorem 6 for (16) and (17), we get

$$E - E^* = -(E - E^*)$$

i.e $E - E^* = 0$ so $E = E^*$, therefore E is normal.

Again satisfy Theorem 6 we get $E = -E$, therefor $E = 0$.

Corollary 5: *If N and M are commuting normal operators and $N = A^*MB$ & $N = B^*MA$, where $A + B$ is cramped unitary operators, then $N = M$.*

Proof: Since

$AN = AA^*MB$ & $BN = BB^*MA$

since A and B are unitary so

$AN = MB$ & $BN = MA$

now 0 doesn't belong to W($A + B$) since $A + B$ is cramped. Thus Theorem 6 is applicable.

Corollary 6: *Let*

$AN = MB$ & $BN = MA$

and

$A^*N = MB^*$ & $B^*N = MA^*$,

where 0 *doesn't belong to* W($A + B$). *If A and B are unitary or N and M are normal, then* $N = M$.

Proof. If N and M are normal then by Theorem 4 we have

$AN^* = M^*B$ & $BN^* = M^*A$

and

$A^*N^* = M^*B^*$ & $B^*N^* = M^*A^*$.

If A and B are unitary, then these equations are also hold since

$NA^* = B^*M$ & $NB^* = A^*M$

and

$NA = BM$ & $NB = AM$.

Define

$$X = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & N \\ M^* & 0 \end{bmatrix}, \quad Y^* = \begin{bmatrix} 0 & M \\ N^* & 0 \end{bmatrix}$$

$$XY^* = \begin{bmatrix} 0 & AM \\ BN^* & 0 \end{bmatrix}, \quad YX = \begin{bmatrix} 0 & NB \\ M^*A & 0 \end{bmatrix}$$

So

$XY^* = YX$.

And W(X) = W($A + B$).

By Corollary 2 we have $Y = Y^*$. Thus $N = M$.

Corollary 7: *If*

$$NN^* = MM^* \quad \& \quad N^*N = M^*M$$

and

$$AN = MB \quad \& \quad BM^* = N^*A,$$

where 0 doesn't belong to W($A + B$)*, then* $N = M$*.*

Proof. Define

$$X = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & N \\ M^* & 0 \end{bmatrix}, \quad Y^* = \begin{bmatrix} 0 & M \\ N^* & 0 \end{bmatrix}$$

$$Y\,Y^* = \begin{bmatrix} NN^* & 0 \\ 0 & M^*M \end{bmatrix}, \qquad Y^*\,Y = \begin{bmatrix} M\,M^* & 0 \\ 0 & N^*N \end{bmatrix}$$

So Y is normal.

$$X\,Y = \begin{bmatrix} 0 & AN \\ BB^* & 0 \end{bmatrix}, \qquad Y^*\,X = \begin{bmatrix} 0 & MB \\ N^*A & 0 \end{bmatrix}$$

So $XY = Y^*X$ and by Corollary 3 we get $Y = Y^*$. Thus $N = M$.

5 Conclusion

We extend Fuglede-Putnam theorem to four bounded linear operatoprs.

Acknowledgement

This research was partially supported by college of science in Diyala University and college of science in Kirkuk University. We are thankful to our colleague Dr. Mohammed Abdollah who provided expertise that greatly assisted the research.

Competing Interests

Author has declared that no competing interests exist.

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